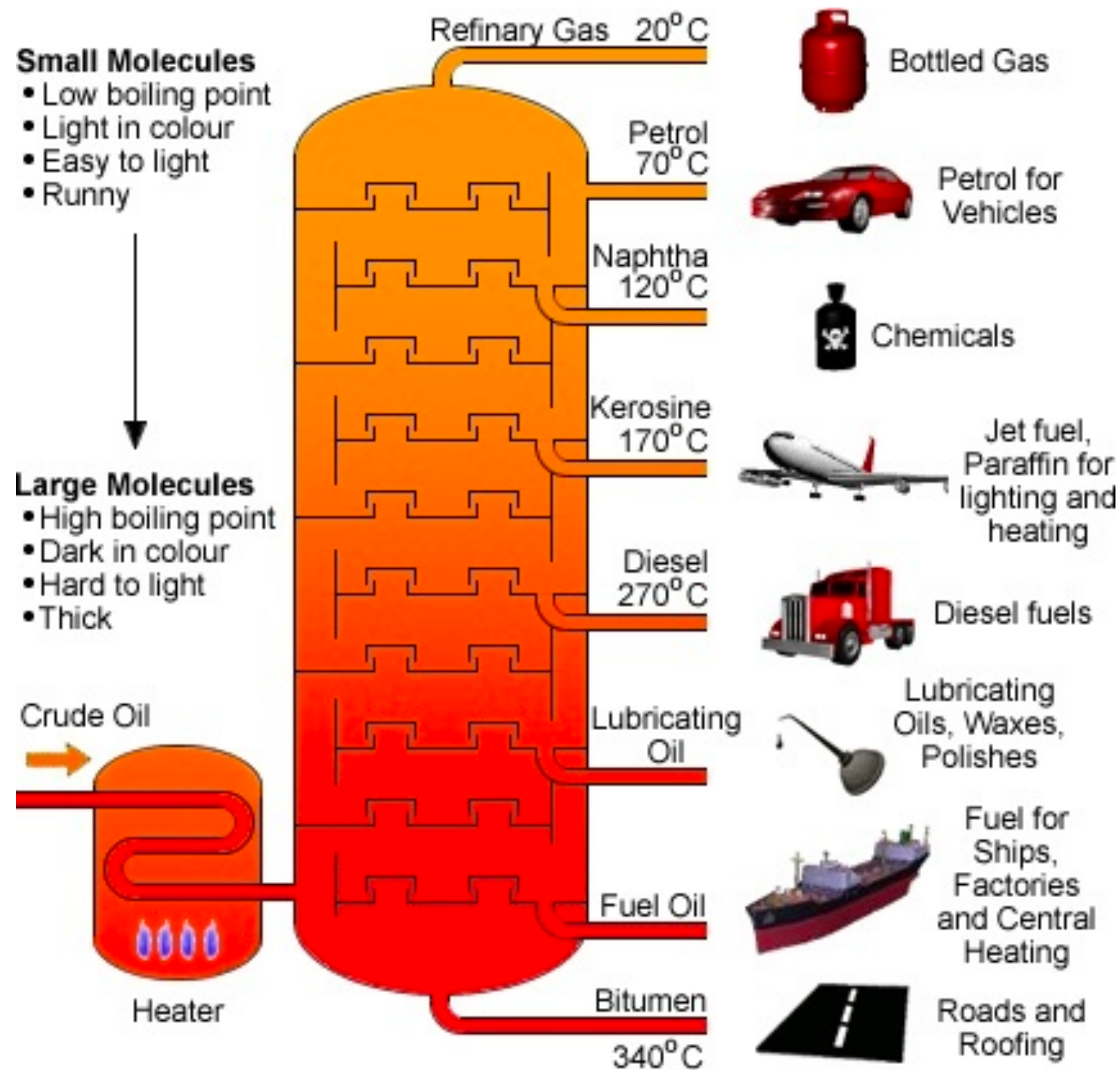


# **Lecture 5**

## **Multicomponent distillation**

# Multicomponent distillation



Fractionating Column

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# Intended Learning Outcome:

1. To calculate minimum number of stages ( $N_{\min}$ ) as a function of desired fractional recovery.
2. To calculate minimum reflux ratio ( $R_{\min}$ )
3. To calculate needed number of stages based on a given reflux ratio.

# Important notation in multicomponent distillation

**Not all component are specified in the distillate and the bottoms**

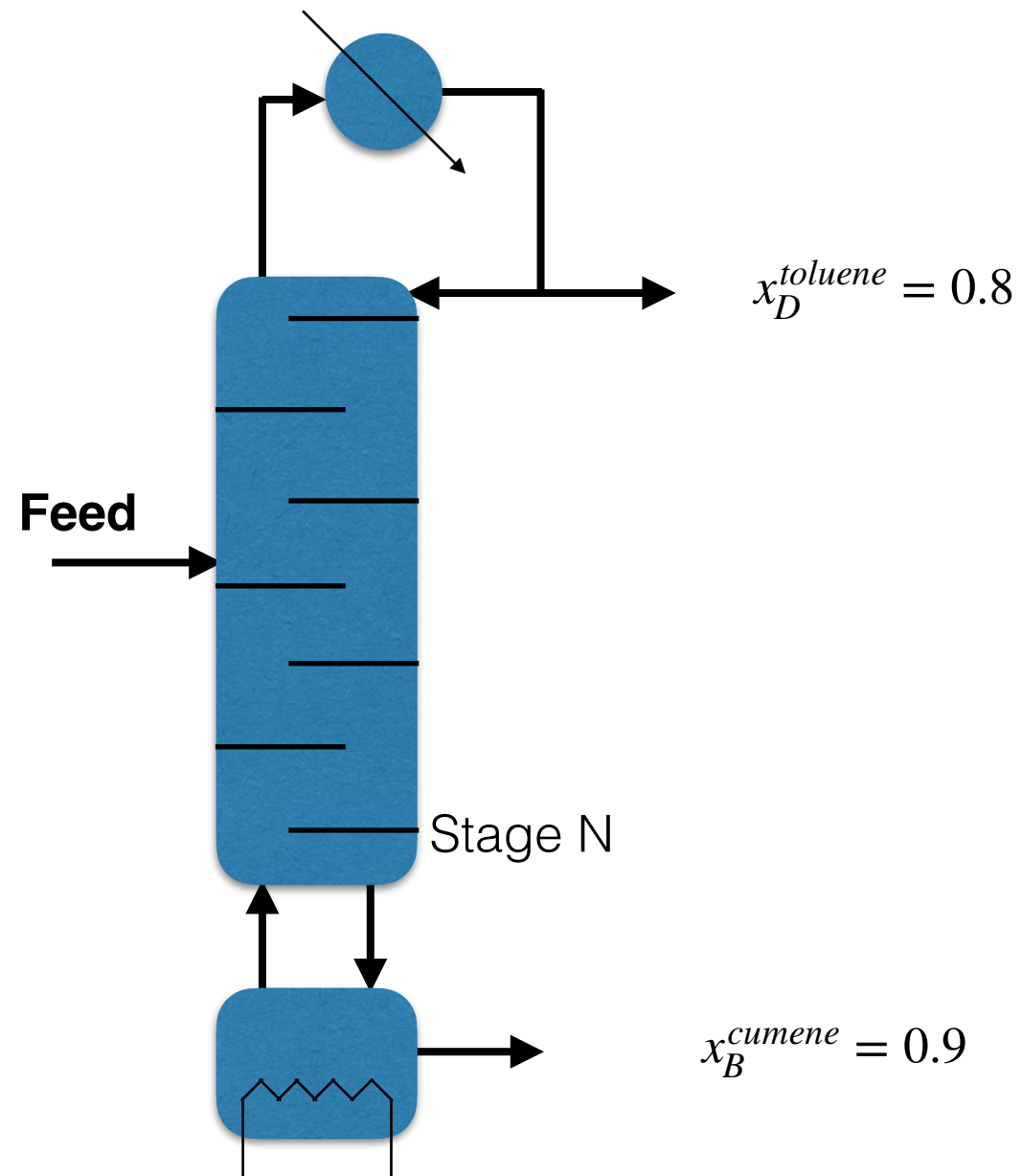
1. Components that are specified are called keys.
2. Components that are not specified are called non-keys (NK).
3. Most volatile of the keys is called light key (LK).
4. Least volatile of the keys is called heavy key (HK).
5. In some cases, we have a light non-key (LNK), when a NK is more volatile than LK.
6. In some cases, we have a heavy non-key (HNK), when a HNK is less volatile than HK.

# Problem Statement

Which component is LK

- A) Cumene
- B) Toluene
- C) Benzene
- D) None of the above

% in feed		
Benzene	80.1 °C	20%
Toluene	110.6 °C	50%
Cumene	152.4 °C	30%

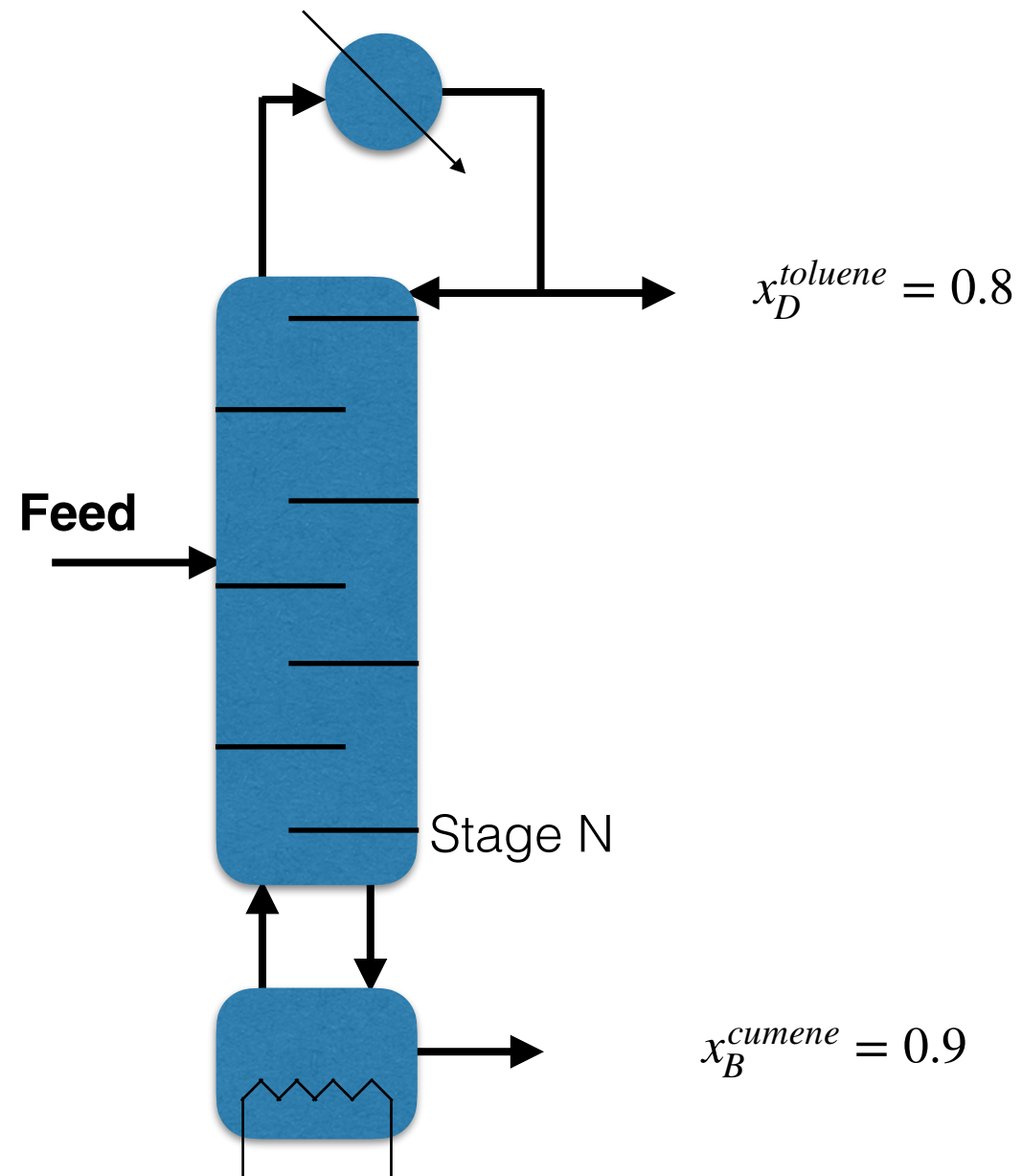


# Problem Statement

Which statement is false

- A) NK = Benzene
- B) NK = LNK
- C) NK = HNK

% in feed		
Benzene	80.1 °C	20%
Toluene	110.6 °C	50%
Cumene	152.4 °C	30%

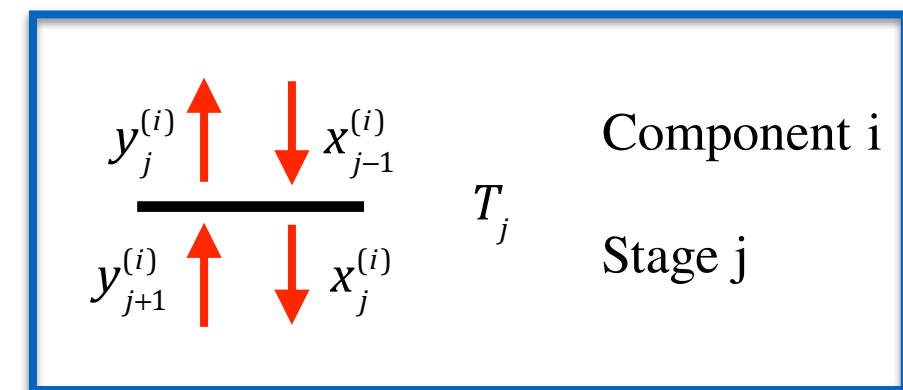
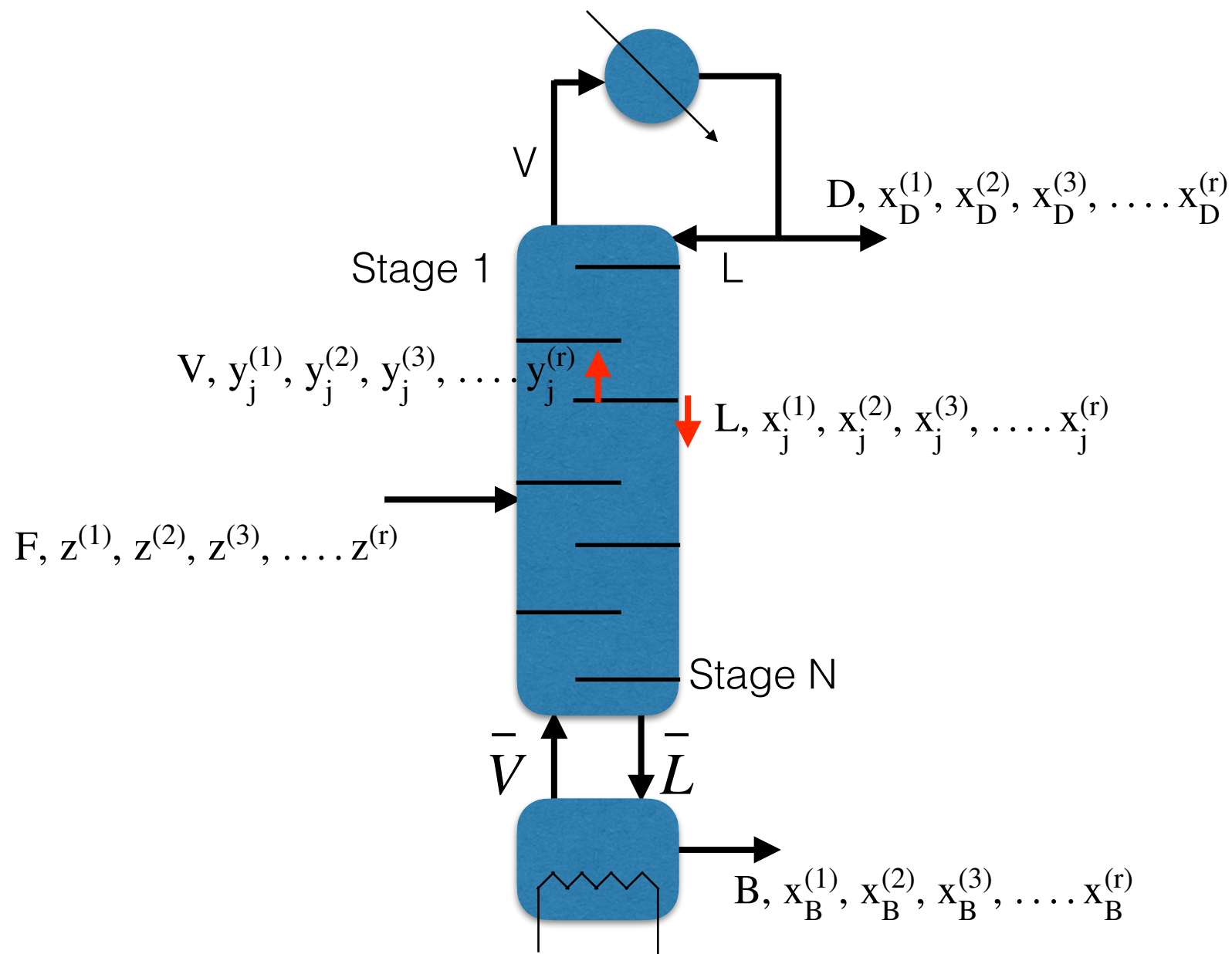


# Multicomponent distillation for r component

## Notation:

Subscript refers to position in distillation column.

Superscript refers to component #



# The concept of fractional recovery, FR

Fractional recovery of component 1 in distillate =  $\frac{\text{Amount of component 1 in distillate}}{\text{Amount of component 1 in feed}}$

$$FR_D^{(1)} = \frac{Dx_D^{(1)}}{Fz^{(1)}}$$

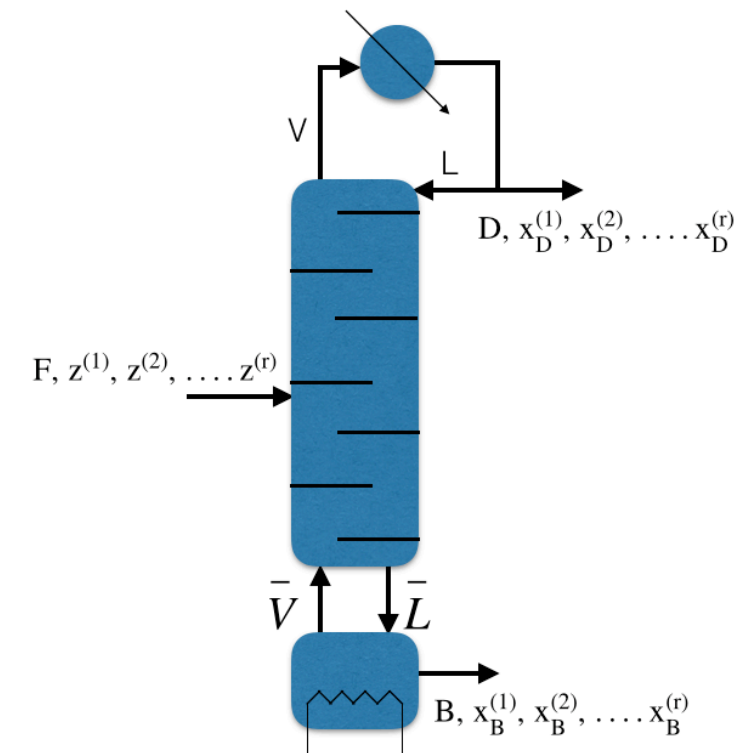
Fractional recovery of component 1 in bottom =  $\frac{\text{Amount of component 1 in bottom}}{\text{Amount of component 1 in feed}}$

$$FR_B^{(1)} = \frac{Bx_B^{(1)}}{Fz^{(1)}}$$

$$FR_D^{(1)} + FR_B^{(1)} = 1$$

$$\frac{FR_D^{(1)}}{FR_B^{(1)}} = \frac{Dx_D^{(1)}}{Bx_B^{(1)}}$$

$$\frac{FR_D^{(1)}}{1 - FR_D^{(1)}} = \frac{Dx_D^{(1)}}{Bx_B^{(1)}}$$





# Minimum number of stages: $N_{\min}$

Condition for  $N_{\min}$  is:  $R \rightarrow \infty, D \rightarrow 0, B \rightarrow 0, F \rightarrow 0$

Can we establish relationship between  $x_D^{(i)}$  and  $x_B^{(i)}$

At stage 1:  $y_1^i = x_D^i$

$y_1^i$  is in equilibrium with  $x_1^i \Rightarrow y_1^i = k_1^{(i)} x_1^i \Rightarrow x_D^i = k_1^{(i)} x_1^i$

Operating equation for stage 1:

$$V = L + D \Rightarrow V = L$$

$$V y_2^i = L x_1^i + D x_D^i \Rightarrow y_2^i = \frac{L}{V} x_1^i + \frac{D}{V} x_D^i \Rightarrow y_2^i = x_1^i$$

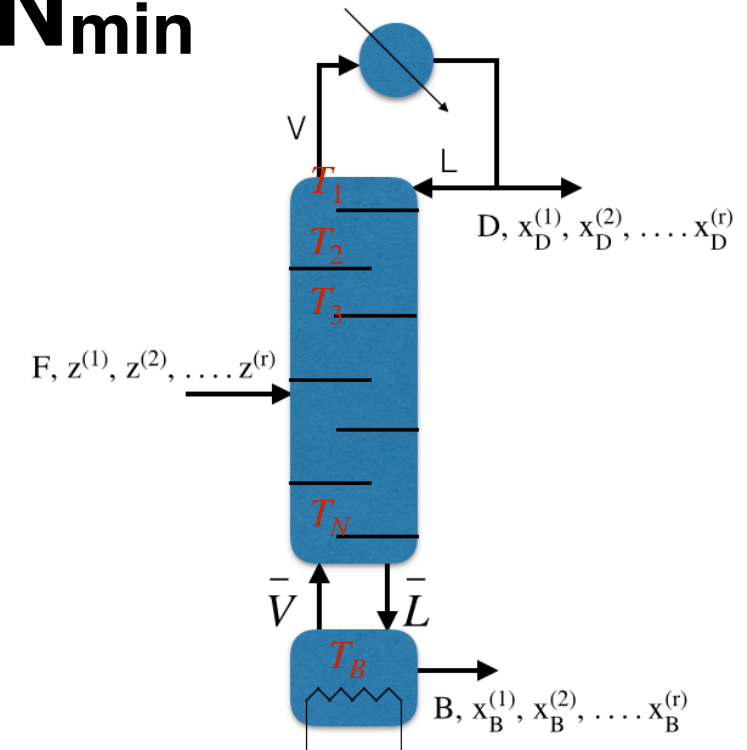
$y_2^i$  is in equilibrium with  $x_2^i \Rightarrow y_2^i = k_2^{(i)} x_2^i \Rightarrow x_1^i = k_2^{(i)} x_2^i$

$$\Rightarrow x_D^i = k_1^{(i)} k_2^{(i)} x_2^i$$

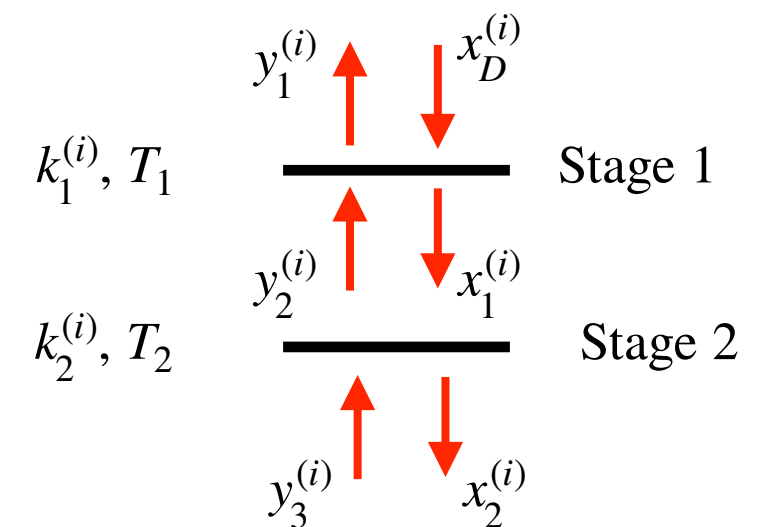
$$\Rightarrow x_D^i = k_1^{(i)} k_2^{(i)} k_3^{(i)} \dots k_{N-1}^{(i)} k_N^{(i)} k_B^{(i)} x_B^i$$

$F=0$ , so no changes at feed plate

$$x_D^j = k_1^{(j)} k_2^{(j)} k_3^{(j)} \dots k_{N-1}^{(j)} k_N^{(j)} k_B^{(j)} x_B^j$$



Component i



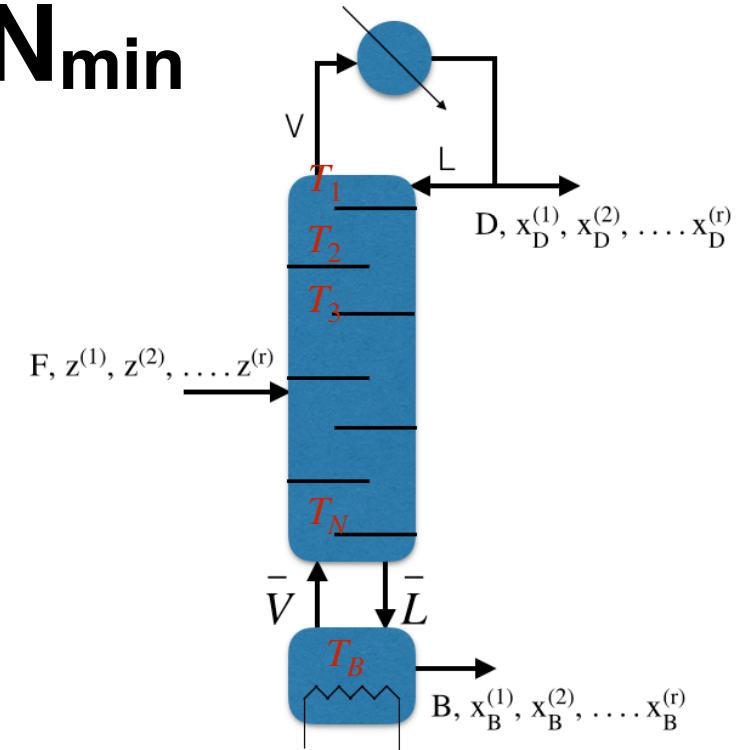
# Minimum number of stages: $N_{\min}$

$$x_D^i = k_1^{(i)} k_2^{(i)} k_3^{(i)} \dots k_{N-1}^{(i)} k_N^{(i)} k_B^{(i)} x_B^i$$

$$x_D^j = k_1^{(j)} k_2^{(j)} k_3^{(j)} \dots k_{N-1}^{(j)} k_N^{(j)} k_B^{(j)} x_B^j$$

Dividing with each other

$$\frac{x_D^i}{x_D^j} = \frac{k_1^{(i)} k_2^{(i)} k_3^{(i)} \dots k_{N-1}^{(i)} k_N^{(i)} k_B^{(i)} x_B^i}{k_1^{(j)} k_2^{(j)} k_3^{(j)} \dots k_{N-1}^{(j)} k_N^{(j)} k_B^{(j)} x_B^j}$$



$$\frac{k_1^i}{k_1^j} = \alpha_1^{ij} \Rightarrow \frac{x_D^i}{x_D^j} = \alpha_1^{(ij)} \alpha_2^{(ij)} \alpha_3^{(ij)} \dots \alpha_{N-1}^{(ij)} \alpha_N^{(ij)} \alpha_B^{(ij)} \frac{x_B^i}{x_B^j}$$

$$\alpha_1^{(ij)} \alpha_2^{(ij)} \alpha_3^{(ij)} \dots \alpha_{N-1}^{(ij)} \alpha_N^{(ij)} \alpha_B^{(ij)} = (\alpha_{average}^{(ij)})^N$$

Similar to the concept of geometric mean

$$\Rightarrow \frac{x_D^i}{x_D^j} = (\alpha_{average}^{(ij)})^N \frac{x_B^i}{x_B^j} \Rightarrow (\alpha_{average}^{(ij)})^N = \frac{x_D^i/x_D^j}{x_B^i/x_B^j} \Rightarrow N = N_{min} = \frac{\ln\left(\frac{x_D^i/x_D^j}{x_B^i/x_B^j}\right)}{\ln(\alpha_{average}^{(ij)})}$$

**Fenske Equation**

# Minimum number of stages: $N_{\min}$

Fenske Equation

$$N = N_{\min} = \frac{\ln\left(\frac{x_D^i/x_D^j}{x_B^i/x_B^j}\right)}{\ln(\alpha_{average}^{(ij)})}$$

We can write in terms of fractional recovery which is usually specified  $FR_D^{(LK)}$  and  $FR_B^{(HK)}$

$$\frac{FR_D^{(i)}}{1 - FR_D^{(i)}} = \frac{Dx_D^{(i)}}{Bx_B^{(i)}}$$

$$N_{\min} = \frac{\ln\left[\left(\frac{FR_D^{(1)}}{1 - FR_D^{(1)}}\right)\left(\frac{FR_B^{(2)}}{1 - FR_B^{(2)}}\right)\right]}{\ln(\alpha_{average}^{(12)})}$$

Usually, component 1 is LK,  
and component 2 is HK

# Explanation

Usually, component 1 is LK, and component 2 is HK

$$N_{\min} = \frac{\ln \left( \frac{x_D^{(1)} x_B^{(2)}}{x_D^{(2)} x_B^{(1)}} \right)}{\ln(\alpha_{\text{average}}^{(12)})}$$

$$N_{\min} = \frac{\ln \left( \frac{Dx_D^{(1)} Bx_B^{(2)}}{Dx_D^{(2)} Bx_B^{(1)}} \right)}{\ln(\alpha_{\text{average}}^{(12)})}$$

$$N_{\min} = \frac{\ln \left( \frac{Dx_D^{(1)} Bx_B^{(2)}}{Bx_B^{(1)} Dx_D^{(2)}} \right)}{\ln(\alpha_{\text{average}}^{(12)})}$$

$$FR_D^{(1)} = \frac{Dx_D^{(1)}}{Fz^{(1)}}$$

$$FR_B^{(1)} = \frac{Bx_B^{(1)}}{Fz^{(1)}}$$

$$\frac{Dx_D^{(1)}}{Bx_B^{(1)}} = \frac{FR_D^{(1)}}{FR_B^{(1)}} = \frac{FR_D^{(1)}}{1 - FR_D^{(1)}}$$

$$\frac{Bx_B^{(2)}}{Dx_D^{(2)}} = \frac{FR_B^{(2)}}{FR_D^{(2)}} = \frac{FR_B^{(2)}}{1 - FR_B^{(2)}}$$

$$N_{\min} = \frac{\ln \left[ \left( \frac{FR_D^{(1)}}{1 - FR_D^{(1)}} \right) \left( \frac{FR_B^{(2)}}{1 - FR_B^{(2)}} \right) \right]}{\ln(\alpha_{\text{average}}^{(12)})}$$

# Fractional recovery of non-keys: Fenske Equation

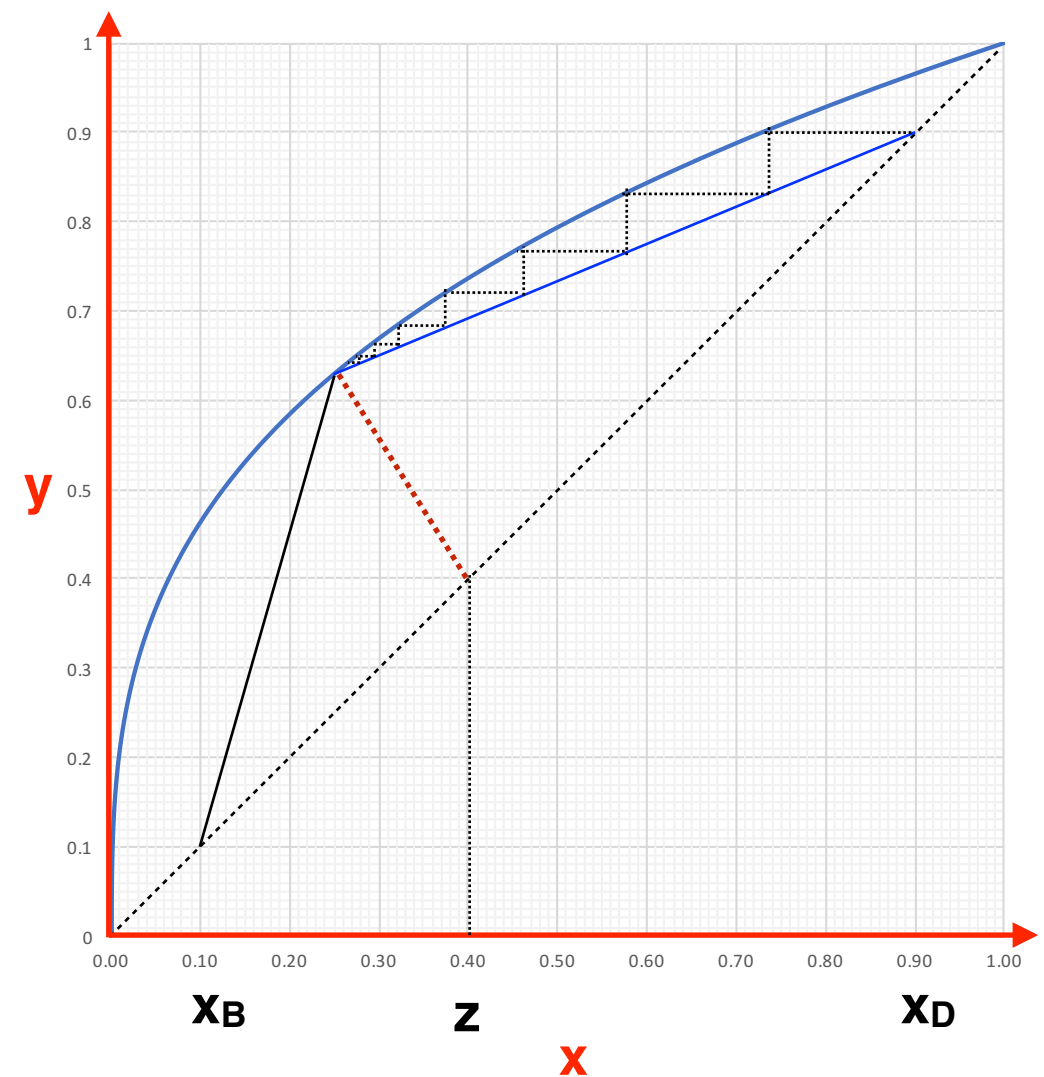
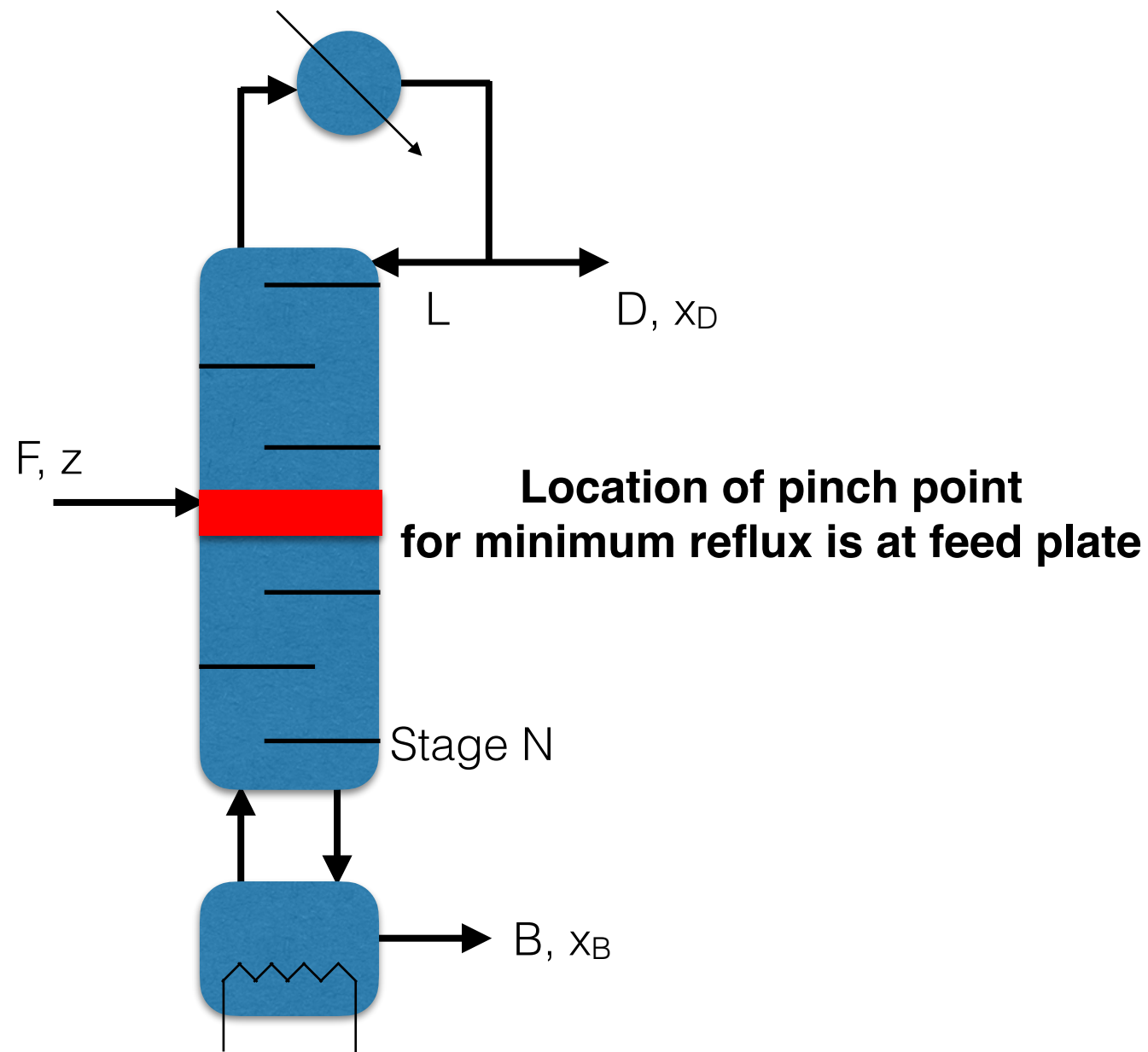
$$N_{\min} = \frac{\ln \left[ \left( \frac{FR_D^{(1)}}{1 - FR_D^{(1)}} \right) \left( \frac{FR_B^{(2)}}{1 - FR_B^{(2)}} \right) \right]}{\ln(\alpha_{average}^{(12)})} \quad \left( \frac{FR_D^{(1)}}{1 - FR_D^{(1)}} \right) \left( \frac{FR_B^{(2)}}{1 - FR_B^{(2)}} \right) = (\alpha_{average}^{(12)})^{N_{\min}}$$

$$FR_D^{(1)} = \frac{(\alpha_{average}^{(12)})^{N_{\min}}}{(\alpha_{average}^{(12)})^{N_{\min}} + \left( \frac{FR_B^{(2)}}{1 - FR_B^{(2)}} \right)}$$

The equation can be adapted for non-keys

$$FR_D^{(3)} = \frac{(\alpha_{average}^{(32)})^{N_{\min}}}{(\alpha_{average}^{(32)})^{N_{\min}} + \left( \frac{FR_B^{(2)}}{1 - FR_B^{(2)}} \right)} \quad FR_B^{(4)} = \frac{(\alpha_{average}^{(41)})^{N_{\min}}}{(\alpha_{average}^{(41)})^{N_{\min}} + \left( \frac{FR_D^{(1)}}{1 - FR_D^{(1)}} \right)}$$

# Recap: minimum reflux ratio in binary distillation



At pinch point, operating and equilibrium line meet

**An analytical equation can be derived in absence of graphical method**

# Analytical method for minimum reflux ratio in binary distillation

At pinch point compositions do not change from stage to stage

$$x_{j-1}^{(1)} = x_j^{(1)} = x_{j+1}^{(1)}$$

$$y_{j-1}^{(1)} = y_j^{(1)} = y_{j+1}^{(1)}$$

Equilibrium

$$y_j^{(1)} = k_j^{(1)} x_j^{(1)}$$

Operating equation in the rectifying section near pinch point

$$V_{min} y_{j+1}^{(1)} = L_{min} x_j^{(1)} + D x_D^{(1)}$$

$$\Rightarrow V_{min} y_j^{(1)} = L_{min} x_j^{(1)} + D x_D^{(1)}$$

Compositions do not change near stage j

$$\Rightarrow V_{min} y_j^{(1)} = L_{min} \frac{y_j^{(1)}}{k_j^{(1)}} + D x_D^{(1)}$$

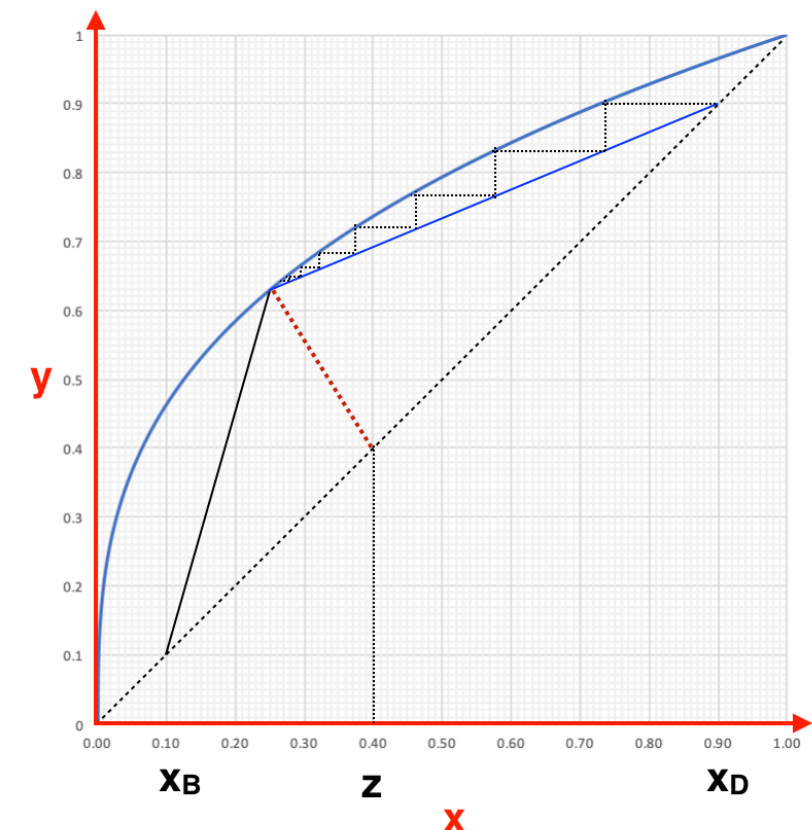
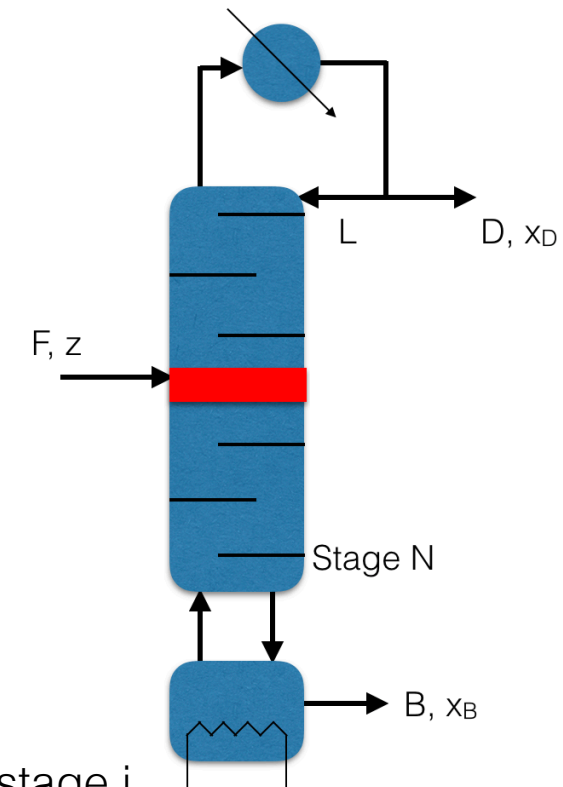
Equilibrium

$$\Rightarrow V_{min} y_j^{(1)} \left( 1 - \frac{L_{min}}{k_j^{(1)} V_{min}} \right) = D x_D^{(1)}$$

$$\Rightarrow V_{min} y_j^{(1)} = \frac{D x_D^{(1)}}{\left( 1 - \frac{L_{min}}{k_j^{(1)} V_{min}} \right)}$$

$$\Rightarrow \sum V_{min} y_j^{(i)} = V_{min} = \sum \frac{D x_D^{(i)}}{\left( 1 - \frac{L_{min}}{k_j^{(i)} V_{min}} \right)}$$

Summation for component 1 and 2; i = 1 and 2



# Analytical method for minimum reflux ratio in binary distillation

Operating equation in the stripping section near pinch point

$$\bar{V}_{min} y_{l+1}^{(1)} = \bar{L}_{min} x_l^{(1)} - B x_B^{(1)}$$

$$\Rightarrow \bar{V}_{min} y_l^{(1)} = \bar{L}_{min} x_l^{(1)} - B x_B^{(1)}$$

$$\Rightarrow \bar{V}_{min} y_l^{(1)} = \bar{L}_{min} \frac{y_l^{(1)}}{k_l^{(1)}} - B x_B^{(1)}$$

Compositions do not change

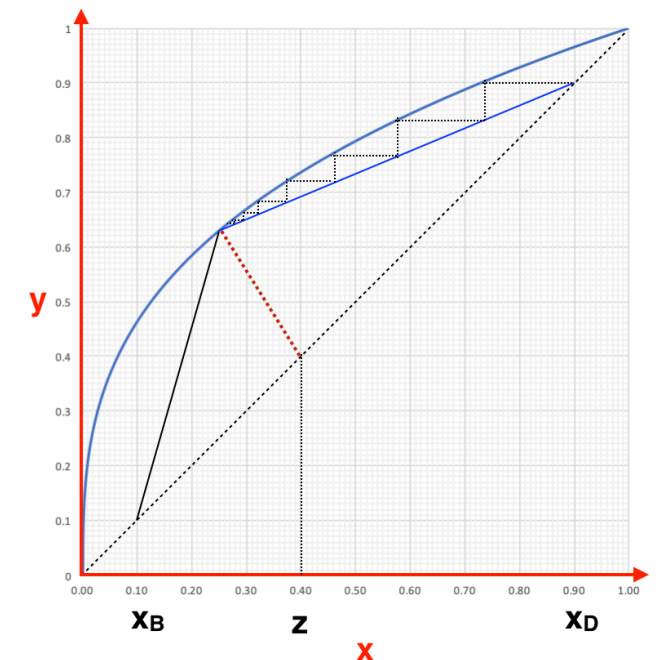
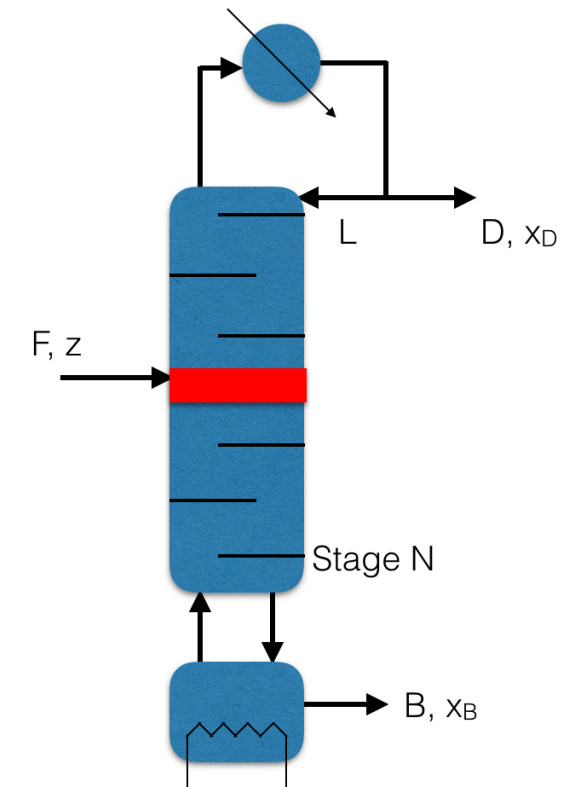
Equilibrium

$$\Rightarrow \bar{V}_{min} = - \sum \frac{B x_B^{(i)}}{\left(1 - \frac{\bar{L}_{min}}{k_l^{(i)} \bar{V}_{min}}\right)}$$

$$V_{min} = \sum \frac{D x_D^{(i)}}{\left(1 - \frac{L_{min}}{k_j^{(i)} V_{min}}\right)}$$

Subtracting

$$V_{min} - \bar{V}_{min} = F(1 - q) = \sum \frac{D x_D^{(i)}}{\left(1 - \frac{L_{min}}{k_j^{(i)} V_{min}}\right)} + \sum \frac{B x_B^{(i)}}{\left(1 - \frac{\bar{L}_{min}}{k_l^{(i)} \bar{V}_{min}}\right)}$$





# Analytical method for minimum reflux ratio in multicomponent distillation

$$F(1 - q) = \sum \frac{Dx_D^{(i)}}{\left(1 - \frac{L_{min}}{k_j^{(i)}V_{min}}\right)} + \sum \frac{Bx_B^{(i)}}{\left(1 - \frac{\bar{L}_{min}}{k_l^{(i)}\bar{V}_{min}}\right)}$$

$$\alpha_j^{(i,HK)} = \frac{k_j^{(i)}}{k_j^{(HK)}}$$

$$\alpha_1^{(i,HK)} = \frac{k_1^{(i)}}{k_1^{(HK)}}$$

$$\Rightarrow F(1 - q) = \sum \frac{Dx_D^{(i)}}{\left(1 - \frac{L_{min}}{\alpha_j^{(i,HK)}k_j^{(HK)}V_{min}}\right)} + \sum \frac{Bx_B^{(i)}}{\left(1 - \frac{\bar{L}_{min}}{\alpha_l^{(i,HK)}k_l^{(HK)}\bar{V}_{min}}\right)}$$

constant relative volatility

$$\alpha_j^{(i,HK)} = \alpha_l^{(i,HK)} = \alpha^{(i,HK)}$$

$$\Rightarrow F(1 - q) = \sum \frac{\alpha^{(i,HK)} Dx_D^{(i)}}{\left(\alpha^{(i,HK)} - \frac{L_{min}}{k_j^{(HK)}V_{min}}\right)} + \sum \frac{\alpha^{(i,HK)} Bx_B^{(i)}}{\left(\alpha^{(i,HK)} - \frac{\bar{L}_{min}}{k_l^{(HK)}\bar{V}_{min}}\right)}$$

Underwood showed that  $\frac{L_{min}}{k_j^{(HK)}V_{min}} = \frac{\bar{L}_{min}}{k_l^{(HK)}\bar{V}_{min}} = \phi$

$$\Rightarrow F(1 - q) = \sum \frac{\alpha^{(i,HK)} Dx_D^{(i)}}{(\alpha^{(i,HK)} - \phi)} + \sum \frac{\alpha^{(i,HK)} Bx_B^{(i)}}{(\alpha^{(i,HK)} - \phi)} = \sum \frac{\alpha^{(i,HK)} (Dx_D^{(i)} + Bx_B^{(i)})}{(\alpha^{(i,HK)} - \phi)} = \sum \frac{\alpha^{(i,HK)} (Fz_i)}{(\alpha^{(i,HK)} - \phi)}$$

# Analytical method for minimum reflux ratio in multicomponent distillation

$$F(1 - q) = \sum_{i=1}^r \frac{\alpha^{(i,HK)}(Fz_i)}{(\alpha^{(i,HK)} - \phi)} \quad \Rightarrow \quad (1 - q) = \sum_{i=1}^r \frac{\alpha^{(i,HK)}z_i}{(\alpha^{(i,HK)} - \phi)}$$

r solution, pick a solution such that  $\alpha^{(1,HK)} < \phi < \alpha^{(r,HK)}$

Understood showed that this is valid only for components which distribute (in distillate as well as bottom)

$$V_{min} = \sum \frac{Dx_D^{(i)}}{\left(1 - \frac{L_{min}}{k_j^{(i)}V_{min}}\right)} \quad \alpha_j^{(i,HK)} = \frac{k_j^{(i)}}{k_j^{(HK)}}$$

$$\Rightarrow V_{min} = \sum \frac{\alpha^{(i,HK)}Dx_D^{(i)}}{\left(\alpha^{(i,HK)} - \frac{L_{min}}{k_j^{(HK)}V_{min}}\right)} \quad \phi = \frac{L_{min}}{k_j^{(HK)}V_{min}} = \frac{\bar{L}_{min}}{k_l^{(HK)}\bar{V}_{min}}$$

$$\Rightarrow V_{min} = \sum \frac{\alpha^{(i,HK)}Dx_D^{(i)}}{(\alpha^{(i,HK)} - \phi)}$$

$$V_{min} = L_{min} + D \quad \Rightarrow \quad L_{min} = V_{min} - D \quad \Rightarrow \quad R_{min} = \frac{L_{min}}{D}$$

# Calculation of Minimum Reflux in Distillation Columns

Three methods for calculating minimum reflux rates in distillation columns fractionating multicomponent mixtures are presented and evaluated. The first method is based on some overlooked

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*Shell Development Company, San Francisco, Calif.*

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*University of California, Berkeley, Calif.*

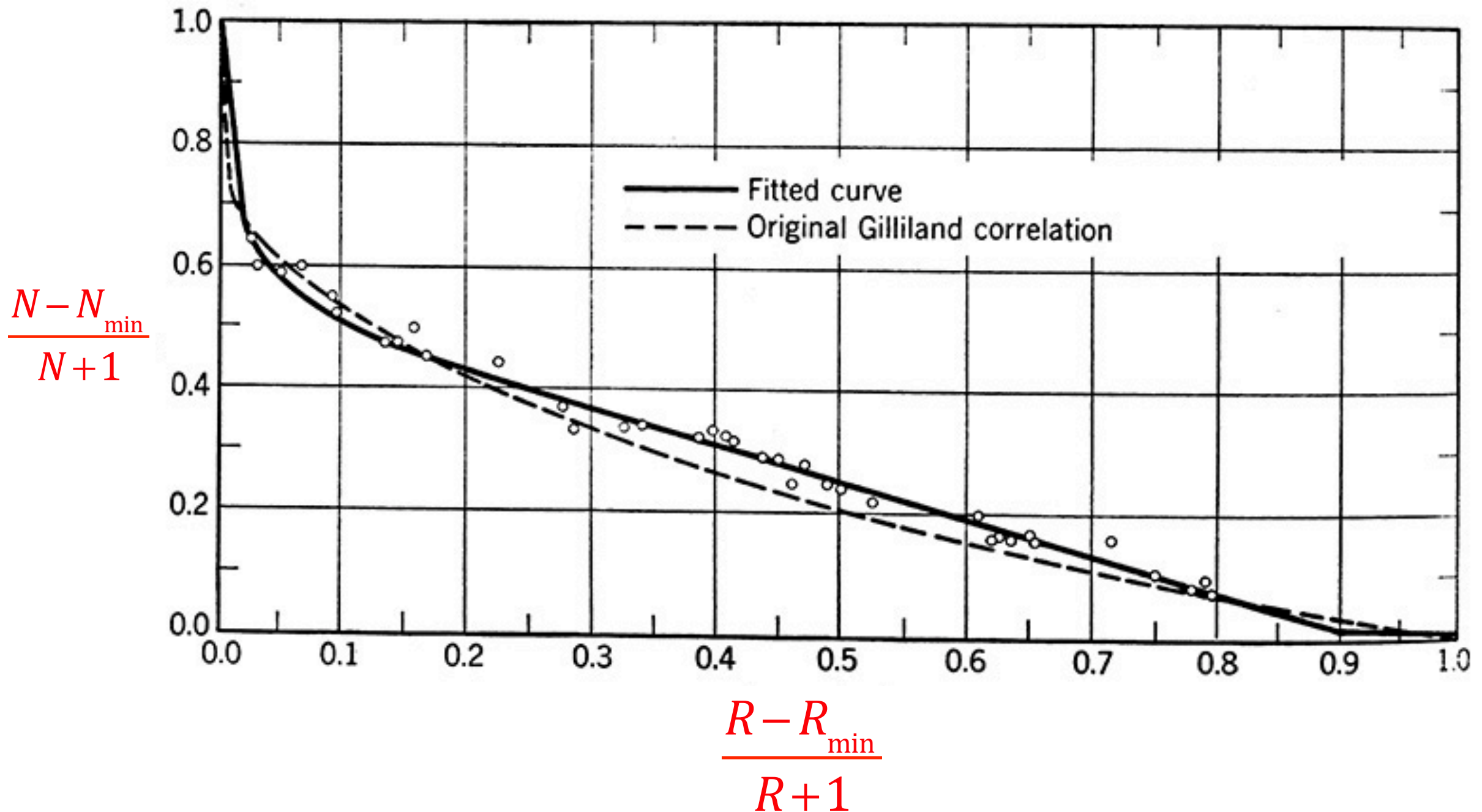
volatility rates are valid. It is also suitable as an approximate method for all column design problems which normally occur. The third method is an adaptation of the Thiele and Geddes plate-

TABLE III. COMPUTATION OF MINIMUM REFLUX

Compn.	$\alpha_{ir}$	Recovery in Top Product, %	Comment	$\Phi_{E,F} = 1.2477$	
				$d(X_i)d$	Eq. 18 $\frac{1.2477 d(X_i)d}{\alpha_{ir} - 1.2477}$
A	10	100	Nondistributing	0.050	0.0071
B	5	100	Nondistributing	0.050	0.0166
C	2.05	94	From Table II	0.094	0.1462
D	2.0	90	Light key	0.270	0.4478
E	1.5	50	From Table II	0.025	0.1237
F	1.0	10	Heavy key	0.030	-0.1511
G	0.9	2	From Table II	0.002	-0.0072
H	0.1	0	Nondistributing	...	...
				$d = 0.512$	$0.5831 = L_{min}$

# Approximate method for multicomponent: Gilliland correlation for number of stages

You can use fitted curve



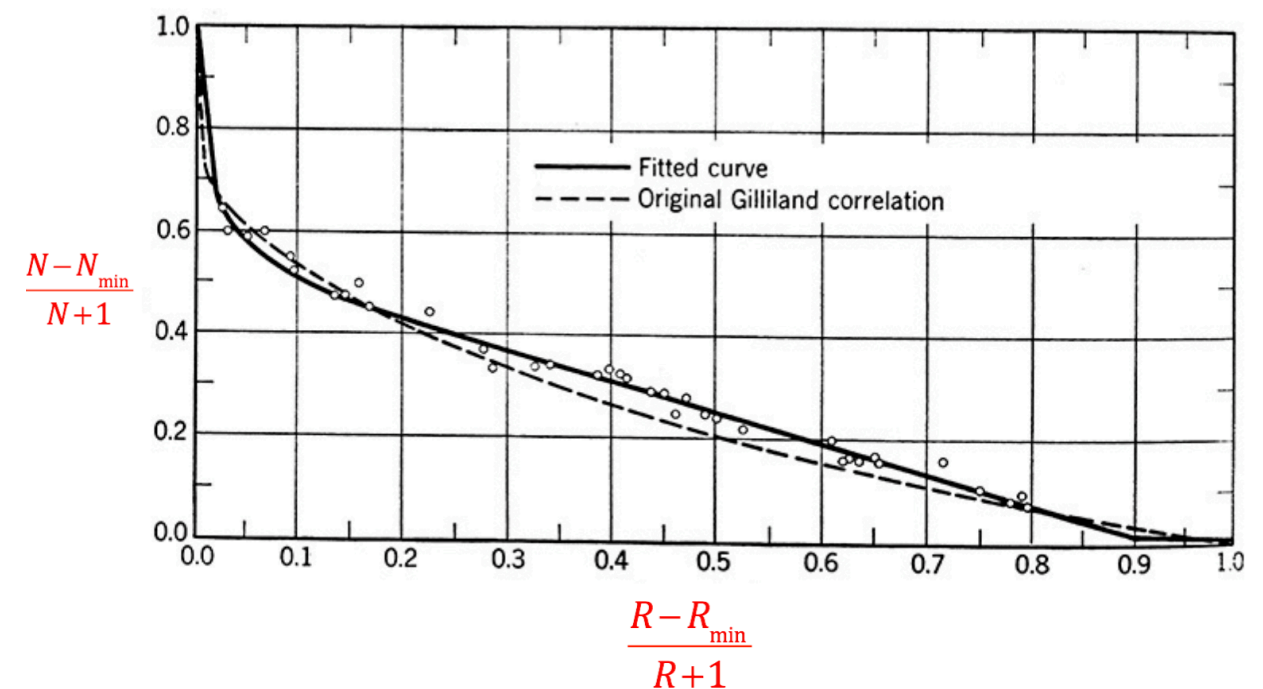
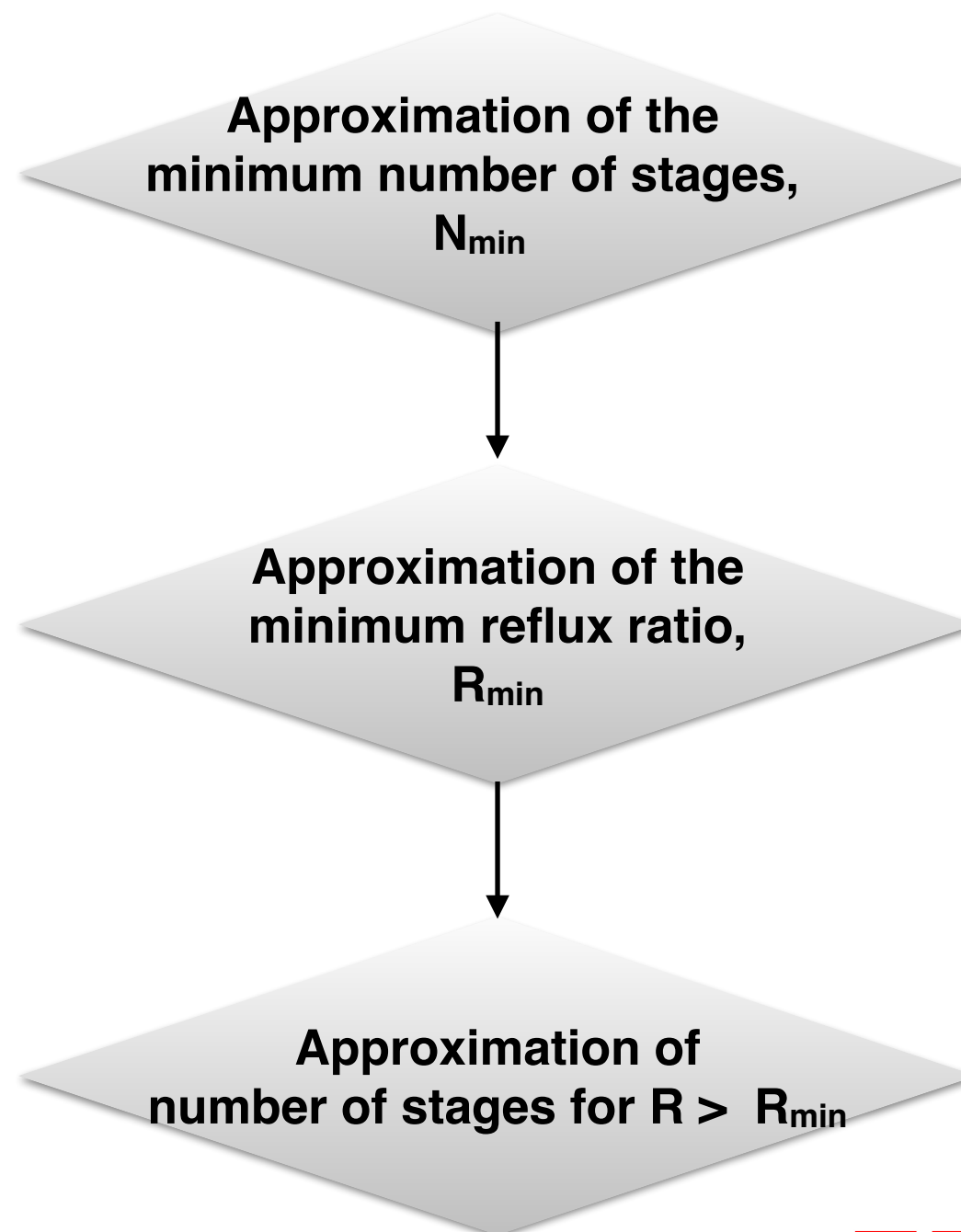
# Optimum feed plate

Empirical equation of Kirkbridge (1944)

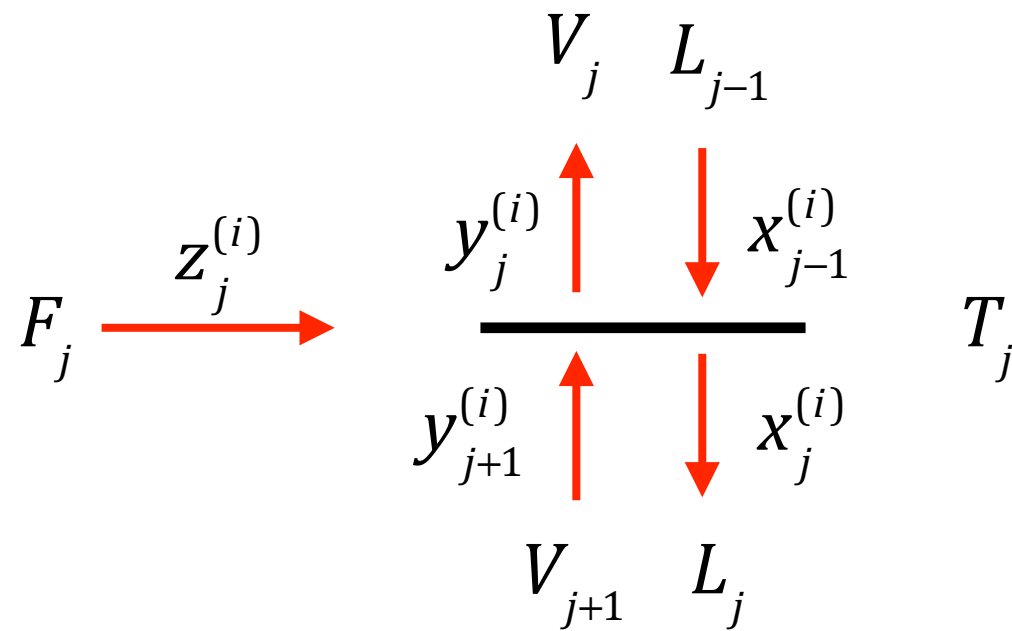
$$\frac{N_{rectifying\_section}}{N_{stripping\_section}} = \left[ \left( \frac{Z^{(HK)}}{Z^{(LK)}} \right) \left( \frac{X_B^{(LK)}}{X_D^{(HK)}} \right)^2 \left( \frac{B}{D} \right) \right]^{0.206}$$

# Approximate method to calculate number of stages

## Fenske-Underwood-Gilliland method



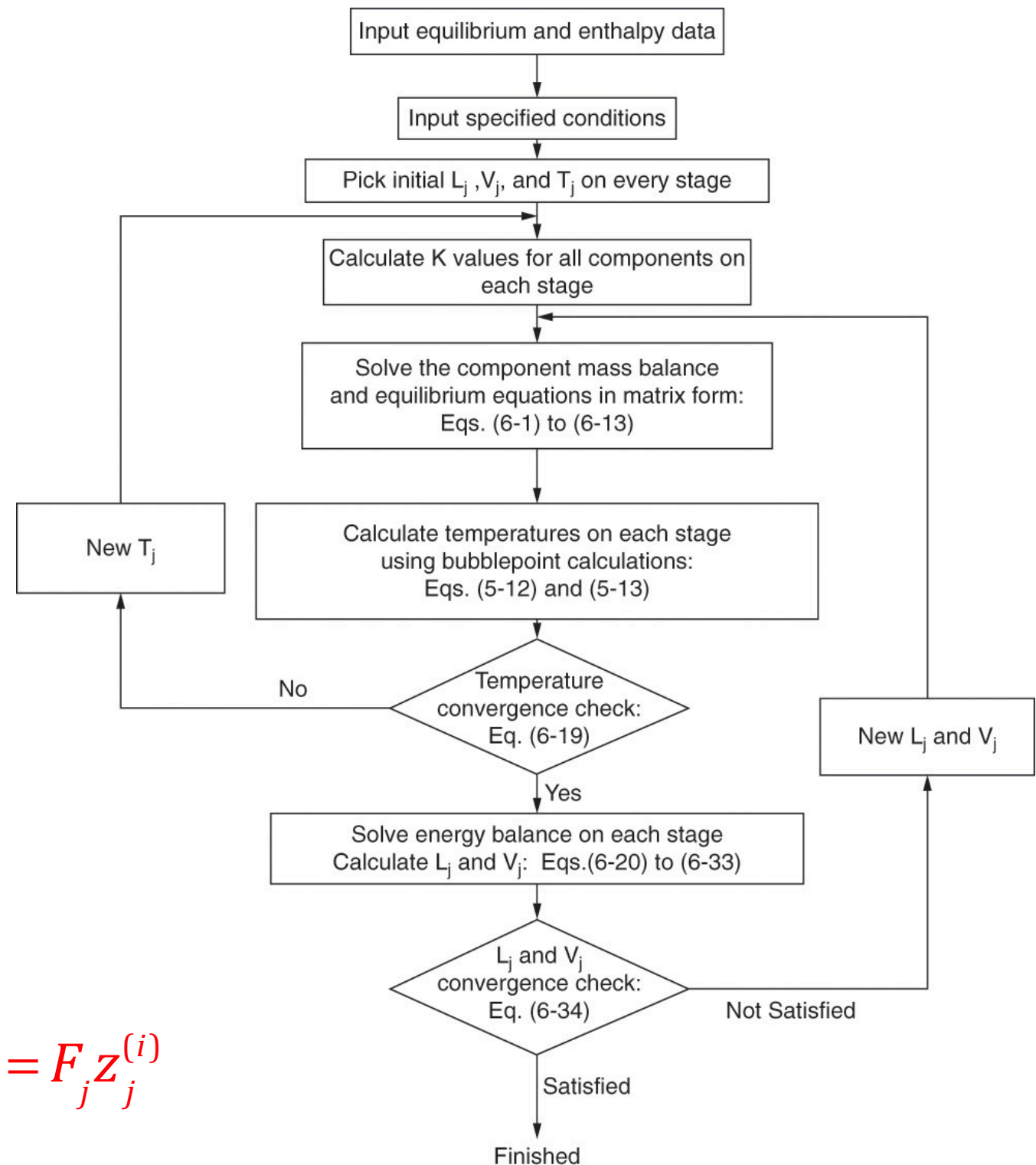
# Exact calculation for multicomponent distillation



$$V_j y_j^{(i)} + L_j x_j^{(i)} - V_{j+1} y_{j+1}^{(i)} - L_{j-1} x_{j-1}^{(i)} = F_j z_j^{(i)}$$

$$y_j^{(i)} = K_j^{(i)} x_j^{(i)}$$

$$V_j K_j^{(i)} x_j^{(i)} + L_j x_j^{(i)} - V_{j+1} K_{j+1}^{(i)} x_{j+1}^{(i)} - L_{j-1} x_{j-1}^{(i)} = F_j z_j^{(i)}$$



# Exercise Problem 1

A three component feed (100 mole/hr, saturated liquid, LNK 10%, LK 55%, HK 35%) is to be separated in a distillation column. Desired recovery for LK in distillate is 99.5%. Assume constant relative volatility. LNK does not go to bottom.

$$x_D^{(LK)} = 0.75 \quad \alpha^{(LNK,LK)} = 4 \quad \alpha^{(HK,LK)} = 0.75$$

1. Calculate B, D, and  $\alpha_{LK,HK}$ ,  $\alpha_{LNK,HK}$ ,  $\alpha_{HK,HK}$ .
2. Calculate minimum number of stages by the Fenske method.
3. Using Underwood equation, calculate possible values of  $\phi$
4. If  $R = 1.2 R_{\min}$ , calculate the number of stages by Gilliland method.

$$N_{\min} = \frac{\ln \left[ \left( \frac{FR_D^{(1)}}{1 - FR_D^{(1)}} \right) \left( \frac{FR_B^{(2)}}{1 - FR_B^{(2)}} \right) \right]}{\ln(\alpha_{\text{average}}^{(12)})} \quad (1 - q) = \sum_{i=1}^r \frac{\alpha^{(i,HK)} z_i}{(\alpha^{(i,HK)} - \phi)}$$

$$0.995 = \frac{D * 0.75}{100 * 0.55} \quad \Rightarrow D = 99.5 * 55 / 75 = 73 \text{ mole/hr} \quad \Rightarrow B = 100 - 73 = 27 \text{ mole/hr}$$

$$0.005 = \frac{27 * x_B^{(LK)}}{100 * 0.55} \quad \Rightarrow x_B^{(LK)} = 0.01 \quad \Rightarrow x_B^{(HK)} = 0.99 \quad \Rightarrow FR_B^{(HK)} = \frac{0.99 * 27}{100 * 0.35} = 76.4 \%$$

LNK does not go to bottom

$$\alpha^{(LK,HK)} = 1 / \alpha^{(HK,LK)} = 1.33$$

$$N_{\min} = \ln(0.995 / 0.005 * 0.764 / 0.236) / \ln(1.33) = 22.7$$



**A three component feed (100 mole/hr, saturated liquid, LNK 10%, LK 55%, HK 35%) is to be separated in a distillation column. Desired recovery for LK in distillate is 99.5%. Assume constant relative volatility.**

$$x_D^{(LK)} = 0.75 \quad \alpha^{(LNK,LK)} = 4 \quad \alpha^{(HK,LK)} = 0.75$$

$$\alpha^{(LNK,HK)} = k_{LNK}/k_{HK} = k_{LNK}/k_{LK} * k_{LK}/k_{HK} = \alpha^{(LNK,LK)} * \alpha^{(LK,HK)} = 4 * 1.33 = 5.32 \quad \alpha_{HK,HK} = 1$$

$$(1 - q) = \sum_{i=1}^r \frac{\alpha^{(i,HK)} z_i}{(\alpha^{(i,HK)} - \phi)} \Rightarrow 0 = \frac{5.32 * (0.1)}{5.32 - \phi} + \frac{1.33 * (0.55)}{1.33 - \phi} + \frac{1 * (0.35)}{1 - \phi}$$

$$q = 1$$

$$\Rightarrow 0 = \frac{0.53}{5.32 - \phi} + \frac{0.73}{1.33 - \phi} + \frac{0.35}{1 - \phi}$$

$$\Rightarrow 0.53(1.33 - \phi) * (1 - \phi) + 0.73(5.32 - \phi)(1 - \phi) + 0.35(5.32 - \phi)(1.33 - \phi) = 0$$

$$\phi = 1.10 \text{ or } 3.98$$

$$1 < \phi < \alpha^{(r,HK)}$$

LNK does not distribute, so not considered

$$1 < \phi < 1.33$$

$$\Rightarrow \phi = 1.10$$

$$V_{min} = \sum \frac{\alpha^{(i,HK)} D x_D^{(i)}}{(\alpha^{(i,HK)} - \phi)}$$

We need to calculate  $x_D$

A three component feed (100 mole/hr, saturated liquid, LNK 10%, LK 55%, HK 35%) is to be separated in a distillation column. Desired recovery for LK in distillate is 99.5%. Assume constant relative volatility.

$$\alpha^{(LK,HK)} = 1.33$$

$$\alpha^{(LNK,HK)} = 5.32$$

$$\alpha_{HK,HK} = 1$$

$$0.236 = \frac{73 * x_D^{(HK)}}{100 * 0.35} \quad x_D \text{ for HK} = 0.11$$

	LNK	LK	HK
alpha (i, HK)	5.32	1.33	1
x <sub>D</sub>	0.14	0.75	0.11

$$\phi = 1.10$$

$$V_{min} = \sum \frac{\alpha^{(i,HK)} D x_D^{(i)}}{(\alpha^{(i,HK)} - \phi)} = 249$$

$$V_{min} = L_{min} + D$$

$$\Rightarrow L_{min} = V_{min} - D = 176$$

$$\Rightarrow R_{min} = \frac{L_{min}}{D} = 2.4$$

$$\Rightarrow R = 1.2 R_{min} = 2.88$$

$$\frac{R - R_{min}}{R + 1}$$

$$\Rightarrow \frac{R - R_{min}}{R + 1} = \frac{0.48}{3.88} = 0.12$$

$$\Rightarrow \frac{R - R_{min}}{R + 1} = 0.12$$

$$\Rightarrow \frac{N - N_{min}}{N + 1} = 0.48$$

$$\Rightarrow \frac{N - 22.7}{N + 1} = 0.48$$

$N = 44.6$  implying 45 stages

